CALIFORNIA INSTITUTE OF TECHNOLOGY

EARTHQUAKE ENGINEERING RESEARCH LABORATORY

THE C. I. T. MARK II ELECTRIC ANALOG TYPE RESPONSE SPECTRUM ANALYZER FOR EARTHQUAKE EXCITATION STUDIES

by

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Pasadena, California
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Division of Engineering

A revision of a report originally issued as the "Sixth Technical Report", Office of Naval Research Contract No. N6 ONR-244, Task Order 25, Project Designation NR-081-095 dated July, 1954.

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TABLE OF CONTENTS

	Page
Abstract	
Introduction	1
Basic Design Principles	2
Time Period Adjustment	5
Zero Damping System	6
Circuit Damping for Repetitive Operation	8
General Layout and Arrangement of Controls	9
Arbitrary Function Generator	12
Time Scale Factor Determination	15
Response Scale Factor Determination	16
Operation and Test of the Spectrum Analyzer	20
Nomenclature	24
List of Figures	26

ABSTRACT

The design of a response spectrum analyzer for earthquake excitation studies is described. Electric analog techniques are used, with a series inductance, capacitance, and resistance circuit forming a direct analog to the mechanical structure. The circuit arrangement permits a determination of system response for a sequence of periods at constant damping. Provision is made for obtaining zero damping in the circuit. An arbitrary function generator of the variable width film-photoelectric cell type is described. The results obtained with the function generator - spectrum analyzer system for a half-sine wave pulse are compared with the mathematically obtained exact answers for the zero damping case, and the accuracy of the system is shown to be satisfactory.

This is a revision of a report originally issued as the "Sixth Technical Report", Office of Naval Research Contract N6 ONR-244, Task Order 25, Project Designation NR-081-095 by the California Institute of Technology dated July, 1954. The new model spectrum analyzer herein described has superseded the model discussed in this preceding report.

INTRODUCTION

The present report has the following objects:

(1) to describe briefly the general design considerations behind the development of the response spectrum analyzer;

(2) to serve as operating instructions for persons using the device; and

(3) to provide details on circuits and instrument arrangement to serve as a guide for checking and maintenance of the analyzer.

In the report these three aspects are treated more-or-less simultaneously, since it is believed that an understanding of the basic design is essential for proper use of the instrument.

A previous report(1) discussed the application of electric analog techniques to the determination of the response spectra of strong motion earthquake acceleration records and presented a large number of such response spectra for typical earthquakes. For that work the general purpose electric analog computer of the California Institute of Technology's Analysis Laboratory was used. Although this general purpose instrument was satisfactory, it was felt that the amount of future computations of this type that would be required might justify the design and construction of a small special purpose instrument for this particular

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(1) Alford, J. L., Housner, G. W., and Martel, R. R., Spectrum Analysis of Strong-Motion Earthquakes, 1st Technical Report, ONR-081-095, Pasadena, California, August 1951.

work. Such a unit, it was thought, might be designed so that the particular measurements required for the response spectra could be made more quickly or more conveniently than on the general purpose computer, and in any event, the continual availability of a special machine would be an advantage. The present report describes the design evolved for this special purpose instrument, and the results which have been obtained with the completed analyzer which is now available for a continuing program of earthquake investigations.

BASIC DESIGN PRINCIPLES

The problem which is to be solved may be stated as follows: given a single-degree-of-freedom system of mass m, spring constant k, and viscous damping c (Fig. 1), for a prescribed base acceleration $\ddot{y}(t)$, find the maximum relative motion $(x-y)_{max}$ between the mass and the base. This corresponds to a determination of the strains set up in a simple building by earthquake ground motions. In particular, a plot of this maximum relative motion versus the undamped natural period, $T_p = 2\pi\sqrt{m/k}$ of the system is desired, for various values of damping expressed as a fraction of critical damping (c/c_c), where $c_c = 2\sqrt{km}$.

As is discussed in detail in previous reports[2], for the particular type of transient exciting forces involved in earthquakes, it is more convenient to plot the relative velocity spectrum $(\dot{x}-\dot{y})_{max}$ rather than

(2) Hudson, D. E., "Response Spectrum Techniques in Engineering Seismology", Proceedings of World Conference on Earthquake Engineering, Earthquake Engineering Research Institute and University of California, Berkeley, 1956.

the relative displacement spectrum, since the velocity spectrum exhibits the essential features of the shape of the spectrum curve in the clearest form. The relative displacements for such earthquake excitations can then be obtained if desired to a satisfactory approximation from the relative velocity spectrum by the relation $(x-y)_{max} \approx \frac{T}{2\pi}(\dot{x}-\dot{y})_{max}$.

The differential equation describing Fig. 1 is:

$$(\ddot{x}-\ddot{y}) + \frac{c}{m}(\dot{x}-\dot{y}) + \frac{k}{m}(x-y) = -\ddot{y}(t) \qquad (1)$$

We are given $\ddot{y}(t)$ and wish to find $(\dot{x}-\dot{y})_{max}$ for various values of the parameters $\left(\frac{c}{m}\right)$ and $\left(\frac{k}{m}\right)$. Considering the complexity of the earthquake exciting forces, and the ranges of the basic parameters involved in typical building structures, something like 300-400 points representing solutions of the differential equation must be obtained to satisfactorily define the response spectrum curves. This large number of points and the very complex nature of the exciting functions suggests that some kind of machine calculation technique should be used. The desired speed of operation and the wide range of system parameters involved indicates an electrical computing system, and the fact that only moderate accuracy is require (2 - 5°/o) suggests an analog type rather than a digital type.

Two basic kinds of electric analog computers would be suitable for the present calculations. The first type sets up a direct mechanical-electrical analogy by constructing a passive electrical circuit whose behavior is described by the same differential equations as the mechanical system. In the second type, the electrical elements are used to carry out the operations indicated by the differential equation. Either method would seem to be satisfactory for the present problem. Since the work already done had used the first method with complete success, it was decided to continue with this passive system.

The advantage of the passive system lies in the fact that for this particular mechanical system the equivalent electric system is a very simple one, and hence a simpler computer results. In particular, the operational type of analog would require a larger number of electronic amplifiers, and would thus be a more complicated device to construct and to maintain.

The basic computer circuit is shown in Fig. 1. The input function is applied as a voltage $E(t)$ which varies in time in the same way as the base acceleration $\ddot{y}(t)$ of the structural system. The differential equation describing the electrical circuit of Fig. 1 is

$$L\ddot{Q} + R\dot{Q} + \frac{1}{C}Q = E(t) \tag{2}$$

where L = inductance, R = resistance, C = capacitance, Q = charge, and $E(t)$ is the applied voltage. A direct comparison of equations (1) and (2) will show the formal analogy between the systems, and it will be seen that the voltage across the resistance in the electrical circuit is analogous to the relative velocity in the mechanical system.

Since rapid operation of the whole computing device was required, it was felt that no recording system, either photographic or pen-writing, should be employed. The most desirable arrangement would result in a display of the response on a cathode-ray tube, which would enable one not only to note the maximum responses rapidly, but would also permit an examination of the whole time response picture for interesting features. This cathode-ray tube display also lends itself directly to a rapid study of system parameter changes. In order to examine transient responses by eye on a cathode-ray tube, it is necessary to repeat the cycle of events at a sufficiently high rate that a stationary pattern is obtained. For the cathode-ray tubes ordinarily used in standard oscilloscopes, a repetition rate of 10 cycles per second is satisfactory, and this rate plus the detail which must be reproduced during each cycle fixes the general frequency limits involved for the computer elements.

Although the simple circuit of Fig. 1 contains the essential elements of the spectrum analyzer, there are a number of additional features that are required in a practical device. These additional features will first be listed, and then the means of attaining them will be discussed.

(1) It must be possible to adjust the circuit parameters in such a way that the natural period goes through a seqqence of values while the damping remains constant.

(2) Various values of damping, including zero damping, must be attainable without otherwise changing the circuit characteristics.

(3) Measurements of the response must be possible. without an undue disturbance of the circuit characteristics.

(4) General requirements of ease of operation and maintenance, stability of operation and calibration, etc., must be met.

TIME PERIOD ADJUSTMENT

The undamped natural period of the electrical circuit of Fig. 1, $T_a = 2\pi\sqrt{LC}$, can be varied by changing either the inductance, L , or the capacitance, C . If, however, either L or C is changed alone, the fraction of critical damping of the circuit would be altered, since the critical damping resistance in such a circuit is $R_c = 2\sqrt{L/C}$. In using the general purpose analog computer of the Analysis Laboratory, it was necessary to compute for each period the required circuit resistance to give the desired damping ratio, and then to adjust the circuit to this value. It was primarily this time-consuming feature that suggested the desirability of a special purpose spectrum analyzer.

In the present instrument, this adjustment is accomplished by changing L and C simultaneously in such a way that the quantity $\sqrt{L/C}$ remains constant. This is done by mechanically coupled switches as indicated in Fig. 1. This procedure has the additional advantage that the period is directly proportional to the inductance, thus giving a linear period scale with standard decade inductors.

ZERO DAMPING SYSTEM

The damping in a passive series LRC circuit such as is shown in Fig. 1 can never be reduced to zero, since there is always a small resistance associated with the inductance. In addition to this difficulty, it is found that even for moderate values of damping, which could be attained in the passive circuit, the resistance R (Fig. 1) becomes so small that the voltage drop across the resistor is difficult to measure. To reduce the damping to zero, an amplifier is employed in such a way that it acts effectively as a negative resistance to cancel the resistance of the rest of the circuit.

The general arrangement of the zero damping amplifier is shown in Fig. 1 and can be explained in simple terms by reference to Fig. 2.

The resistance R of the series circuit is replaced by a combination of a resistance in series with an amplifier. If the gain of the amplifier is just unity, then the voltage at the amplifier output would be equal to the voltage at the input, and the total voltage drop from A to B would be just that due to the damping resistor R . By increasing the gain of the amplifier above unity, the voltage at B can be increased and hence the voltage drop from A to B can be decreased. The amplifier thus acts like a negative resistance, and can be adjusted to balance out the damping resistor and the resistance of the rest of the circuit. This same amplifier arrangement makes it possible to introduce a measuring resistor, R_0 , to improve the accuracy of measurement of the voltage drop across the resistance.

To adjust the system to zero damping, a periodic test signal introduces an "initial displacement" in the system once per cycle. The resulting free oscillations can be visually observed on the screen of a cathode-ray oscilloscope, as indicated in Fig. 3. The gain of the amplifier is adjusted until the oscillations remain at a constant amplitude, as in Fig. 3(c), thus indicating zero damping. Various amounts of positive damping can then be introduced into the system if desired, either by changing the damping resistance R or by decreasing the amplifier gain. The circuit constants are such that $\sqrt{L/C} = 200\pi$, and hence the total resistance for critical damping is $R_c = 2\sqrt{L/C} = 400\pi = 1255$ ohms.

As shown in the schematic diagram of the complete Function Generator - Spectrum Analyzer system in Fig. 4, the positive-gain zero-damping amplifier is attained by a combination of two negative-gain amplifiers. This makes it possible to use standard commercial type amplifiers, in this case type USA-3 chopper-stabilized units manufactured by the George A. Philbrick Company. The general overall characteristics of the amplifier combination as employed in the spectrum analyzer are:

(1) Positive gain, adjustable from 1 to 1.6

(2) Negligible phase shift from 0 - 10,000 cyc/sec

(3) Flat response (5%) from 0 - 10,000 cyc/sec

(4) Input impedance of 1 megohm

(5) Output impedance of 1/2 ohm.

CIRCUIT DAMPING FOR REPETITIVE OPERATION

As mentioned above, it is desirable to be able to visually observe the system response so that the maximum values of the response can be quickly noted. The transient input function is applied to the system 10 times per second, so that a stationary pattern of input or response signal can be produced on the cathode-ray oscilloscope screen. This requires that the circuit response be brought periodically to rest at the end of each cycle of operation in readiness for the next force application. This is accomplished by replacing the input function generator for a portion of each cycle by a resistor which is large enough to give somewhat greater than critical damping in the circuit. During the latter portion of each cycle of operation, the circuit oscillations are thus damped out, bringing the system to rest ready for the next cycle. An electronic switch which is synchronized with the input function signal is set so that this large damping resistor is in the circuit for approximately the last one-third of each cycle. The earthquake input function itself must therefore be limited to two-thirds of a cycle; i.e., the time scales must be selected in such a way that the total time duration of the earthquake corresponds to approximately $(2/3)(1/10) = 0.0667$ sec. Since several hundred peaks of earthquake ground motion may need to be included in this time, the reason for a frequency response flat to some 10,000 cyc/sec is evident. The same electronic switch which operates this damping circuit also provides signals for timing the zero damping test, and for synchronizing the cathode-ray oscilloscope.

GENERAL LAYOUT AND ARRANGEMENT OF CONTROLS

The photographs of Figs. 5, 6 and 7 will give a general idea of the mechanical design and construction of the Spectrum Analyzer unit. A complete electrical circuit diagram is given in Fig. 8.

Referring to Fig. 5, the controls of the Spectrum Analyzer may be briefly described as follows. The two upper controls are the decade switches for the 100 discrete period settings. The numbers on the period switches should be divided by 10,000 to get the actual analog period in seconds. For example, a decade switch setting of $20 + 7 = 27$ would give an analog period of 0.0027 sec.

Considering now the lower controls from left to right, the first one is the zero damping test switch. For normal operation of the Spectrum Analyzer, this switch should be in the "operate" position. In the "zero" position the damping test described above is automatically carried out, and pictures of the kind shown in Fig. 3 will be seen on the output oscilloscope. In this position the circuit damping can be adjusted to zero, as in Fig. 3(c), regardless of the setting of the damping resistor R on the "damping selector" switch. In the "test" position a decaying oscillation will be seen on the oscilloscope, corresponding to the value of R set on the "damping selector" switch. If this "damping selector" switch is set to zero, the same response will be obtained on both "zero" and "test" settings. The next switch marked "damping calibration" adjusts the gain of the zero-damping amplifier, and is used to adjust the circuit damping to zero with the "damping test" switch set at "zero". The next switch marked "damping selector" introduces into the analog circuit standard values of damping of 0, 2.5, 5, 10, 15, and 20 per cent of critical damping. The decay rate corresponding to each of these standard damping values can be observed on the output oscilloscope by setting the "damping test" switch to the "test" position. If intermediate values of damping are desired, the "damping calibration" control can be varied, while the "damping test" switch is in the "test" position, to give any desired decay rate. The ratios of successive amplitudes of oscillation corresponding to this desired damping factor can easily be calculated and measured on the output oscilloscope. The switch position marked "ext" is to be used when the inductance, capacitance, and resistance analog elements are supplemented by external elements to extend the scales or to interpolate period readings. The external connections for accomplishing this are described later in this report.

The "mode selector" switch selects the output parameter to be observed on the oscilloscope. In the "input" position the earthquake ground acceleration as produced by the Function Generator may be observed. The "acceleration" position gives an oscilloscope signal proportional to the relative acceleration between the ground and the mass. Similarly, the switch positions marked "velocity" and "displacement" give the relative motions between the ground and the mass. The switch position marked "velocity" thus gives directly the Relative Velocity Response Spectrum, S_v , as usually defined.

If the absolute acceleration of the mass is desired, as it might be to check acceleration measurements or calculated lateral force coefficients, a special connection inside of the analyzer cabinet can be used. By connecting the output oscilloscope to the red terminal at the right front top of the chassis, with the "mode selector" switch in the displacement position, the absolute acceleration of the mass can be observed. The black terminal inside the cabinet is an input function signal, which is useful if both input and response are to be simultaneously observed on a dual-beam oscilloscope. This black terminal will give the ground acceleration input signal for any position of the "mode selector" switch.

The output terminals at the lower right corner of the front panel go to the oscilloscope input, the oscilloscope common or ground terminal, and the oscilloscope external synchronization input. There are two multiple-connector plugs at the rear of the analyzer chassis. The seven-terminal cable connects the Spectrum Analyzer to the Function Generator unit. The twelve-terminal connector makes available external connections to the inductance, capacitance, and resistance in the main analog circuit, as shown in Fig. 8. In this way, additional external elements can be added to extend the scales of the analyzer or to interpolate between period settings. When using these external elements in this way, the "damping selector" switch should be in the "ext" position.

ARBITRARY FUNCTION GENERATOR

The arbitrary Function Generator which is used with the Response Spectrum Analyzer is a modification of a type developed by the Analysis Laboratory and used in the work described in the previous report, Ref. 1.

Various methods have been used to generate arbitrary input functions for electric analog computing work. One method which has been widely used, the "photoformer" method, employs an opaque template having the form of the desired function. This template is placed in front of a cathode-ray tube and the beam is driven to follow the pattern. A second method makes use of a variable-width film in conjunction with a light source and photocell. This second method was selected for the work because of the complexity of the input function and the accuracy desired. By the use of a 10 in. diameter circular track, a record length of over two feet is available, which enables one to reproduce in considerable detail even very complex functions such as the earthquake acceleration record of Fig. 16(b). It was thought that it would not be possible to reproduce such records in the small size required for photoformer generation and retain all small details that might have an effect on system response.

The general features of the Function Generator are shown in the block diagram of Fig. 4, and in the photographs of Figs. 9 and 10. A complete circuit diagram of the Function Generator is given in Fig. 11.

The film record itself is made on the special plotting table shown in Fig. 12. The earthquake ground acceleration record drawn to a suitable scale is wrapped around a drum which is slowly rotated by an electric motor around a vertical axis. The curve is manually traced by a follower mechanism which is connected through a selsyn system to a shutter, thus exposing a photographic negative which rotates along with the drum. In this way, a variable-width film trace is produced as indicated in Fig. 13, where the overall slit width is seen to be equal to a constant plus twice the acceleration function. A similar slit system, along with a light source and a photocell is then used in the Function Generator to reproduce the original ground acceleration curve.

The dimensions of the standard film disk are shown in Fig. 14. By limiting the width of the record as shown in Fig. 14, the photocell system will operate in a linear region. It is a good idea to run a record of a known test function on the Function Generator from time to time to insure that proper adjustment for linearity has been maintained.

The earthquake ground acceleration record should be limited to about two-thirds the circumference of the disk, since about one-third of each cycle is used to damp out the system response in preparation for the next cycle of operation as described above.

An advantage of a film disk record of the present type is that direct photographic reproductions can easily be made from the record itself.

The film records are made on standard Kodalith Ortho film, starting with a 11" x 14" sheet which is trimmed roughly to size before mounting it on the plotting table. All photographic operations can be carried out with illumination from a Dupont S-55X safelight filter. The film developing is done in a standard two-solution Litho developer. This developer consists of one part "A" solution, one part "B" solution, and two parts water to make approximately 32 liquid oz. This developer does not last long after mixing, and a fresh mixture is used for each record. The film is developed in a tray with slight agitation for a time which is 45 seconds longer than required for a black outline to appear on both sides of the film. The film is then fixed for 10 minutes in a standard paper fix solution consisting of one part concentrated fixer and one part water.

The film disk record is rotated in the Function Generator at 10 revolutions per second by a synchronous motor. The same rotating disk which carries the film also carries a small metal plug which energizes a variable reductance element once per revolution, thus producing an electric pulse. This timing pulse is used to synchronize the various elements of the analyzer system, such as the damping circuits and the output oscilloscope.

The light source in the Function Generator is an automobile headlight bulb which is operated at 6 to 8 volts. The lamp power supply must be very well filtered so that the ripple voltage is less than 1°/o. A 0.025 in. wide slit is placed between the film and the phototube. A type 929 phototube with an output voltage of approximately 0.1 volts is used. The phototube preamplifier has a gain of approximately 100, and a frequency response which is flat from 2 - 70,000 cycles per second. The power amplifier is a standard commercially available unit manufactured by the George A. Philbrick Company (Type USA-3). It is a chopper stabilized amplifier with a frequency response flat from 0 - 10,000 cycles per second, with negligible phase shift below 8000 cycles per second. The output impedance is about 1/2 ohm. The voltage output level from the power amplifier is of the order of 10 volts, and the overall noise level through the whole system is about 5 millivolts. The Spectrum Analyzer analog circuits are usually operated at a voltage level of about 2 volts.

The photograph in Fig. 10 shows an internal view of the Function Generator with the drive motor and record disk assembly removed. The front panel view of Fig. 9 shows the controls. The upper control, marked "damping duration" varies the fraction of the cycle during which the response is damped out. The middle control marked "sync. phase" adjusts the point in the cycle at which the large damping is introduced. This control will vary this damping point over a range of 180°. If the earthquake record occupies less than 180° of the film record disk, it will not matter how it is placed on the motor drive disk, since it will always be possible to alter the position at which the damping is introduced to come at an appropriate place in the cycle. If the earthquake spreads over more than half the disk, it will be necessary to position the record on the motor drive disk so that the timing pulse comes in proper relation to the start of the earthquake. The control marked "drive amplitude" changes the output voltage of the generator.

In Fig. 15 is shown a photograph of the complete Function Generator - Spectrum Analyzer - Oscilloscope system as ordinarily used.

TIME SCALE FACTOR DETERMINATION

The relations between the record spacing on the film record in the Function Generator and the actual natural period of the electric circuit in the Spectrum Analyzer fix the time factors for the analog computation. The time factor N is defined as the ratio between the prototype time in the mechanical system and the equivalent analog time in the electrical system.

For example, if an input record having an actual time duration of 15 seconds were reproduced on 180° of the film record disk, then, since the disk rotates at 10 revolutions per second, the time factor would be:

$$N = \frac{15 \text{ sec.}}{\left(\frac{1}{10}\frac{\text{sec}}{\text{rev}}\right)\left(\frac{180^{\circ}}{360^{\circ}}\right)\text{ rev.}} = 300$$

If now this record were to be used in the Spectrum Analyzer with a period decade setting of 20 + 7 = 27, then the actual mechanical prototype period being calculated would be

$$T_p = \frac{(300)(27)}{10,000} = 0.81 \text{ seconds.}$$

RESPONSE SCALE FACTOR DETERMINATION

It is convenient to operate the analog circuits on a considerably faster time scale than the prototype system. The relations between the response factors for the mechanical and the electrical systems may be obtained by writing the differential equations for each system in terms of the same real time, and by directly comparing the corresponding terms.

For the mechanical system, equation (1) can be put in the form:

$$\ddot{z} + 2\omega_p\left(\frac{c}{c_c}\right)\dot{z} + \omega_p^2 z = -\ddot{y}(t) = A_o f(t) \qquad (3)$$

where $z = (x-y)$, ω_p = natural frequency in radians per second of mechanical system, and A_0 is the magnitude scale of the ground acceleration. The dots refer in all cases to differentiations with respect to real prototype time t .

For the electrical system, the time scale is changed so that if t_a is the time in the analog system:

$$t_a = \frac{t}{N} \tag{4}$$

The natural frequency and the natural period of the analog electrical system thus become:

$$\omega_a = N\omega_p \quad ; \quad T_a = \frac{T_p}{N} \tag{5}$$

Equation (2) describing the electrical system can now be re-written, using a prime to indicate differentiation with respect to analog time, t_a , in the following way:

$$Q'' + 2\omega_a\left(\frac{c}{c_c}\right)Q' + \omega_a^2 Q = \frac{E_0}{L} f(Nt_a) \tag{6}$$

Noting now that $\dfrac{d}{dt_a} = \dfrac{d}{dt}\cdot\dfrac{dt}{dt_a} = N\dfrac{d}{dt}$, equation (6) becomes:

$$N^2\ddot{Q} + 2\omega_a\left(\frac{c}{c_c}\right)N\dot{Q} + \omega_a^2 Q = \frac{E_0}{L} f(t)$$

or

$$\ddot{Q} + 2\omega_p\left(\frac{c}{c_c}\right)\dot{Q} + \omega_p^2 Q = \frac{E_0}{LN^2} f(t) \tag{7}$$

Equations (3) and (7) can now be directly compared, since both the mechanical and the electrical equations have been expressed in terms of the same real prototype time t .

For relative displacements, comparing corresponding terms in equations (3) and (7) gives:

$$\frac{\omega_p^2 z}{\omega_p^2 Q} = \frac{A_o f(t)}{\frac{E_o}{LN^2} f(t)}$$

$$z = \frac{A_o LN^2}{E_o} Q = \frac{A_o Q L}{E_o} \frac{\omega_a^2}{\omega_p^2}$$

but $\omega_a^2 = \frac{1}{LC}$, so $z = \frac{A_o Q}{E_o} \frac{1}{C} \frac{1}{\omega_p^2}$

Also we have $\frac{Q}{C} = E_c$ = voltage across capacitor, and $\frac{1}{\omega_p^2} = \frac{T_p^2}{4\pi^2}$

so finally;

$$z = (x-y) = \frac{T_p^2}{4\pi^2}\left(\frac{E_c}{E_o}\right) A_o \qquad (8)$$

For relative velocities, comparing corresponding terms in equations (3) and (7):

$$\frac{2\omega_p\left(\frac{c}{c_c}\right)\dot{z}}{2\omega_p\left(\frac{c}{c_c}\right)\dot{Q}} = \frac{A_o f(t)}{\frac{E_o}{LN^2} f(t)}$$

$$\dot{z} = \frac{A_o LN^2}{E_o} \dot{Q}$$

The voltage E_R across the measuring resistor is $E_R = R_o Q' = R_o N \dot{Q}$, so:

$$\dot{z} = \frac{A_o NL}{E_o} \cdot \frac{R_o N \dot{Q}}{R_o} = A_o\left(\frac{E_R}{E_o}\right) \frac{NL}{R_o}$$

Since $\omega_a = \frac{1}{\sqrt{LC}}$, this becomes:

$$\dot{z} = A_o\left(\frac{E_R}{E_o}\right) \frac{1}{\omega_p} \cdot \frac{1}{\sqrt{LC}} \cdot \frac{L}{R_o} = \frac{T_p}{2\pi}\left(\frac{E_R}{E_o}\right) \frac{\sqrt{L/C}}{R_o} A_o$$

For the particular circuit constants used,

$$\sqrt{L/C} = 200\pi \text{ , and } R_o = 1000 \text{ ohms ,}$$

and we obtain finally:

$$\dot{z} = (\dot{x} - \dot{y}) = 0.1\, T_p \left(\frac{E_R}{E_o}\right) A_o \tag{9}$$

For relative accelerations, in the same way, we have:

$$\frac{\ddot{z}}{\ddot{Q}} = \frac{A_o f(t)}{\frac{E_o}{LN^2} f(t)} \quad ; \quad \ddot{z} = \frac{A_o}{E_o} LN^2 \ddot{Q}$$

but the voltage across the inductor is given by

$$E_L = LQ'' = LN^2\ddot{Q}$$

so finally,

$$\ddot{z} = (\ddot{x} - \ddot{y}) = \left(\frac{E_L}{E_o}\right) A_o \tag{10}$$

The absolute acceleration $\ddot{x}$ is obtained by adding the relative acceleration to the input ground acceleration, which can easily be done in the analog circuit as indicated in Fig. 1.

Equations (8), (9) and (10) give the fundamental relationships between the measured electrical quantities and the desired mechanical quantities. Note that in each case the measured voltages enter into the equation as a voltage ratio, and hence no absolute voltage calibration of the oscilloscope is required. It is only necessary to associate a known deflection on the oscilloscope with a known peak acceleration on the input record.

As an example of the above scale transformations, suppose that the Relative Velocity Response Spectrum values, S_V , are to be determined from an earthquake ground acceleration film disk record made with a time constant N = 378. With the period decade switches set at

20 + 2 = 22, the actual prototype period to be plotted on the final spectrum curves would be:

$$T_p = \frac{(378)(22)}{10,000} = 0.832 \text{ seconds} .$$

Let us suppose that the gain of the oscilloscope amplifier is set so that a particular acceleration peak in the input record which has an acceleration value of 0.28 g, corresponds to 4 divisions on the oscilloscope screen. The "mode selector" switch is then set to "velocity", and, with the damping set at the desired value, the maximum peak response read on the oscilloscope is, say, 3.5 divisions. From equation (9), we would obtain:

$$S_v = \dot{z}_{max} = (\dot{x} - \dot{y})_{max} = (0.1)(0.832 \text{ sec.})\left(\frac{3.5}{4}\right)(0.28\, g.)\left(32.2\, \frac{ft/sec.^2}{g.}\right)$$

$$S_v = 0.656 \, \frac{ft}{sec} .$$

This would be the plotted ordinate on the maximum relative velocity response spectrum curve.

OPERATION AND TEST OF THE SPECTRUM ANALYZER

In Fig. 16 are shown typical input and response records as photographed from the screen of the cathode-ray oscilloscope. Figure 16(a) is a half-sine test signal which was made by a regular film disk on which an accurately plotted half-sine function had been photographically reproduced. Figure 16(b) shows a typical earthquake ground acceleration-time record as reproduced by the Function Generator. This particular photograph is the first 10 seconds of the El Centro earthquake of May 18, 1940, N-S component. In Fig. 16(c) is shown a typical analog response to the earthquake excitation of Fig. 16(b) for a specific period and damping. In this particular case, the maximum response occurred at near the beginning of the earthquake after only two or three cycles. This is not in general to be expected; the maximum value could occur at any time in the duration of the earthquake.

To compare the results obtained on the new C. I. T. Mark II Spectrum Analyzer with past calculations, certain film disk records from past investigations were re-run. In Fig. 17, the solid line is the 0.20 damping S_v spectrum for the El Centro earthquake of May 18, 1940, N-S component, as determined during the original damped spectrum studies made on the general purpose electric-analog computer of the California Institute of Technology (ref. 1). The circled points were obtained using the original film disk record and the first model Spectrum Analyzer described in the original version of the present report. The points marked with triangles were obtained using the original film disk and the new Mark II Spectrum Analyzer and Function Generator. The variations between these three different calculations can easily be accounted for by reasonable errors in reading the cathode-ray oscilloscope amplitudes.

In Fig. 18, the solid curve is the undamped velocity spectrum as determined analytically for a half-sine pulse, for which a mathematical solution can be easily obtained. If t_0 is the pulse duration time, a is the peak pulse acceleration, and T is the period of the system to which the pulse is applied, the velocity during the pulse is given by:

$$\dot{z}(t) = \frac{\frac{a}{\pi} t_o}{1 - \frac{t_o^2}{\pi^2 T^2}} \left(\cos \frac{\pi t}{t_o} - \cos \frac{t}{T} \right) \quad ; \quad 0 < t < t_o$$

and the velocity after the pulse is:

$$\dot{z}(t) = \frac{\frac{2a}{\pi} t_o}{1 - \frac{t_o^2}{\pi^2 T^2}} \cos \frac{t_o}{2T} \quad ; \quad t > t_o$$

The curve in Fig. 18 gives for each period the maximum relative velocity, which for some periods occurs during the pulse, and for others after the pulse. The curve is computed for the specific case of a pulse for which $t_o = 1$ sec. and $a = 0.1$ g. The film record for the pulse was made by photographically reproducing an accurately plotted half-sine wave on a standard size disk. The photograph of Fig. 16(a) shows this half-sine pulse as reproduced by the Function Generator as an input to the Spectrum Analyzer. The points marked with triangles show the results obtained with the new Mark II system. Note that this comparison gives an evaluation of the overall accuracy of the whole process under relatively severe conditions. The small deviations indicated in Fig. 18 can be accounted for by reasonable errors in reading the cathode-ray oscilloscope amplitudes.

One possible source of error in the period scale lies in the fact that the values of the inductances change slightly with current, so that at widely different analyzer levels corrections to the period readings might sometimes be called for. The analyzer as usually employed does not involve a sufficiently wide input voltage level range to require such corrections.

It is believed that accuracies of the order indicated in Figs. 17

and 18 are entirely adequate for the engineering applications contemplated for the Spectrum Analyzer system.

NOMENCLATURE

m	mass of single-degree-of-freedom mechanical system
k	linear spring constant of single-degree-of-freedom mechanical system
c	viscous damping in mechanical system
c_c	critical damping in mechanical or electrical system
$(\frac{c}{c_c})$	fraction of critical damping in mechanical or electrical system
T	period of single-degree-of-freedom system, seconds
T_p	period of mechanical prototype, seconds
T_a	period of electrical analog circuit
ω_p	natural frequency of prototype mechanical system, radians per second
ω_a	natural period of electrical analog system, radians per second
t	real prototype time
t_a	equivalent analog time
t_o	time duration of half-sine pulse
x	absolute displacement of mass in single-degree-of-freedom mechanical system
y	absolute displacement of ground
$z = (x - y)$	relative displacement between ground and mass
$\ddot{y}(t)$	ground acceleration, a function of time $= A_o f(t)$
A_o	magnitude scale of ground acceleration, g's
Q	electric charge in analog circuits
L	electric inductance
C	electric capacitance
R	damping resistance in electric circuit
R_o	measuring resistor for velocity determination

R_c resistance for critical damping

$E(t)$ applied voltage, proportional to ground acceleration

$$E_o f(t) \Rightarrow A_o f(t)$$

E_o magnitude scale of applied voltage

E_c voltage across capacitance

E_R voltage across resistance

E_L voltage across inductance

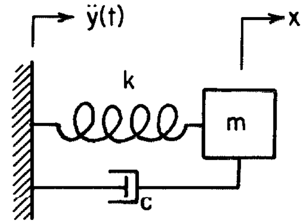
N time scale factor

S_v Maximum Relative Velocity Response Spectrum, ft/sec, defined in Fig. 1

a peak value of half-sine acceleration pulse

LIST OF FIGURES

1. Elements of the Response Spectrum Analyzer
2. Zero-Damping Circuit for the Response Spectrum Analyzer
3. Appearance of Zero-Damping Test for Different Amplifier Gain Settings
4. Schematic Diagram of Function Generator - Spectrum Analyzer System
5. Front Panel View of Spectrum Analyzer
6. Interior View of Spectrum Analyzer
7. Bottom View of Spectrum Analyzer
8. Complete Electrical Circuit Diagram for the Spectrum Analyzer
9. Front Panel View of Function Generator
10. Interior View of Function Generator
11. Complete Circuit Diagram of Function Generator
12. Plotting Table for Function Generator Film Disk Record
13. Transformation of Input Function to Variable-Width Film Trace
14. Dimensions of the Standard Film Disk Record for the Function Generator
15. Photograph of Complete Function Generator - Response Spectrum-Oscilloscope System
16. Typical Input and Response Records Photographed from Oscilloscope Screen
17. Comparison of Response Spectrum Determinations for Earthquake Ground Motion
18. Comparison of Response Spectrum Determinations for Half-Sine Pulse

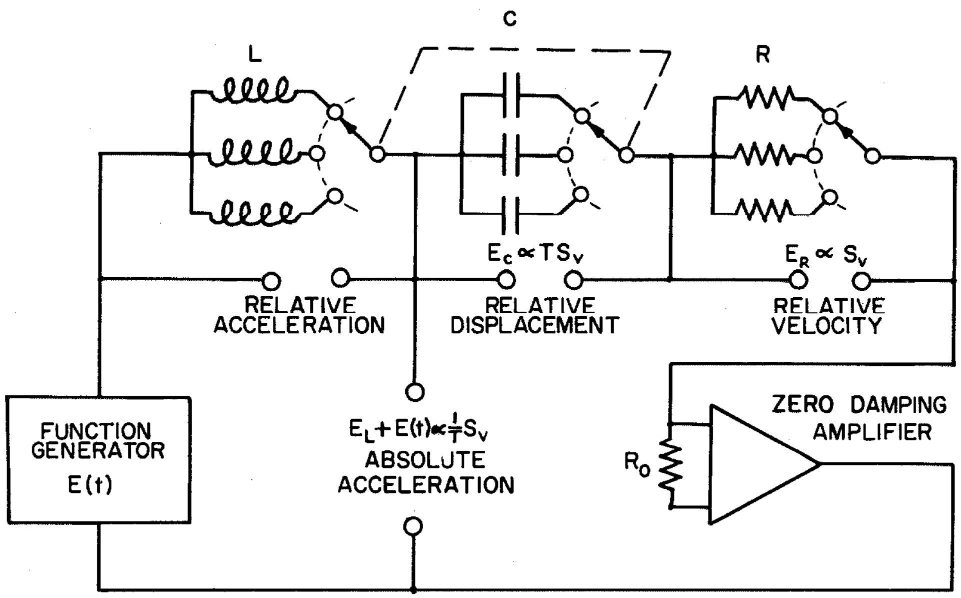


$$Z = (x - y)$$
$$T = 2\pi\sqrt{\frac{m}{k}}$$
$$C_c = 2\sqrt{km}$$

STRUCTURAL SYSTEM

$$\ddot{z} + \frac{4\pi}{T}\left(\frac{c}{c_c}\right)\dot{z} + \left(\frac{2\pi}{T}\right)^2 z = -\ddot{y}(t)$$

$$S_v = (\dot{x}-\dot{y})_{max} = \left[\int_0^t \ddot{y}(\tau) e^{-\frac{2\pi}{T}\left(\frac{c}{c_c}\right)(t-\tau)} \sin \frac{2\pi}{T}(t-\tau)\,d\tau\right]_{max}$$



ELECTRIC ANALOG.

Fig. 1 Elements of the Response Spectrum Analyzer

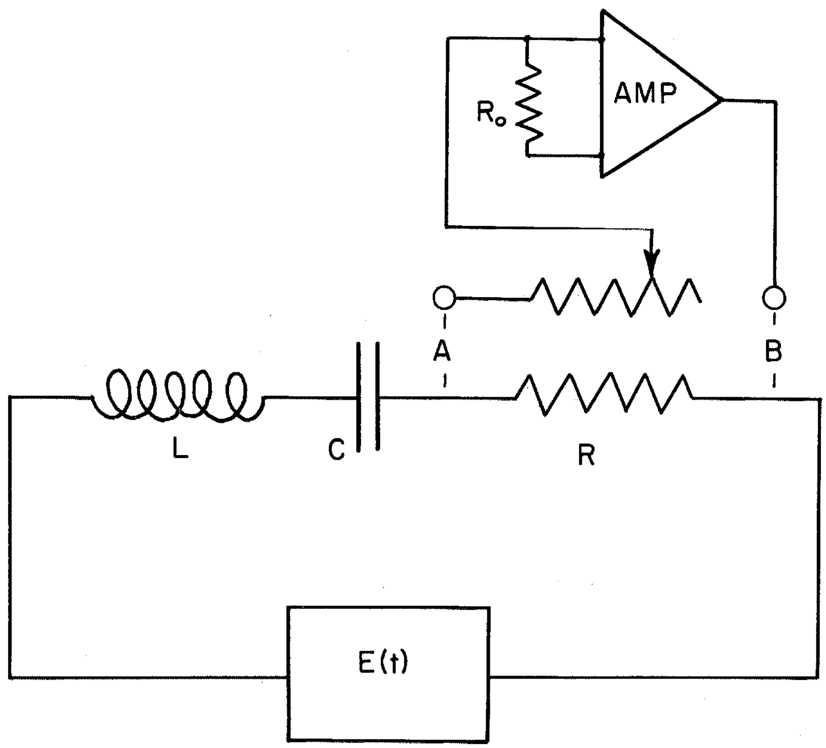


Fig. 2 Zero-Damping Circuit for the Response Spectrum Analyzer

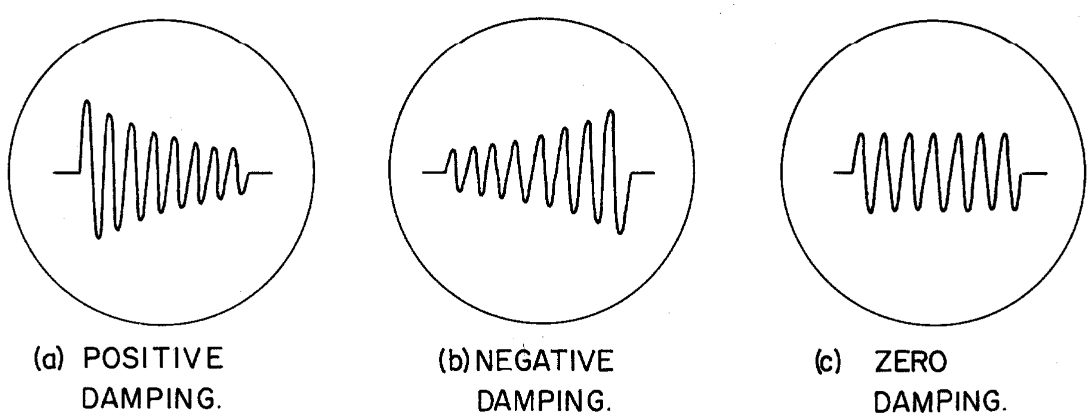


Fig. 3 Appearance of Zero-Damping Test for Different Amplifier Gain Settings

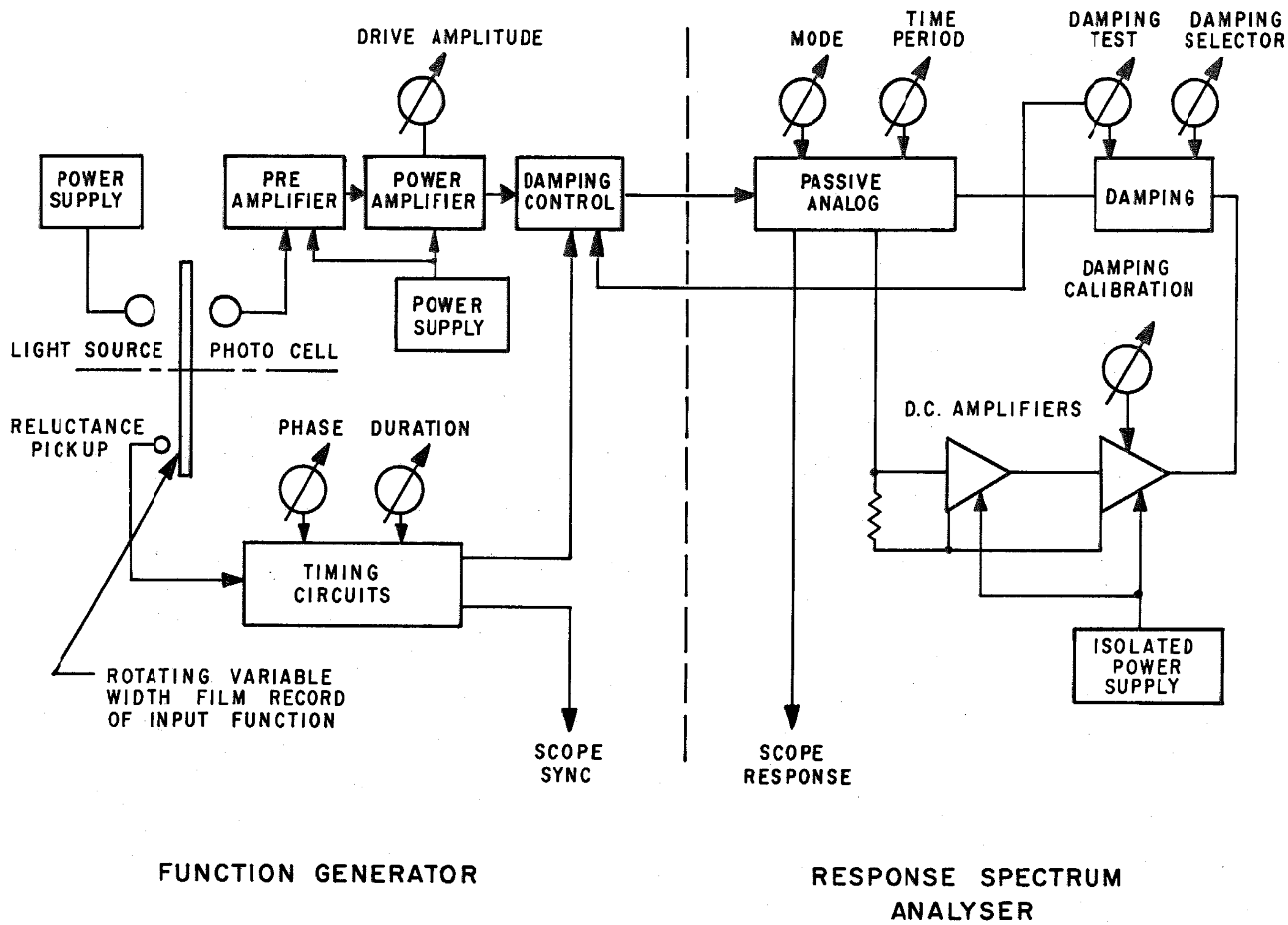


Fig. 4 Schematic Diagram of Function Generator-Spectrum Analyzer System

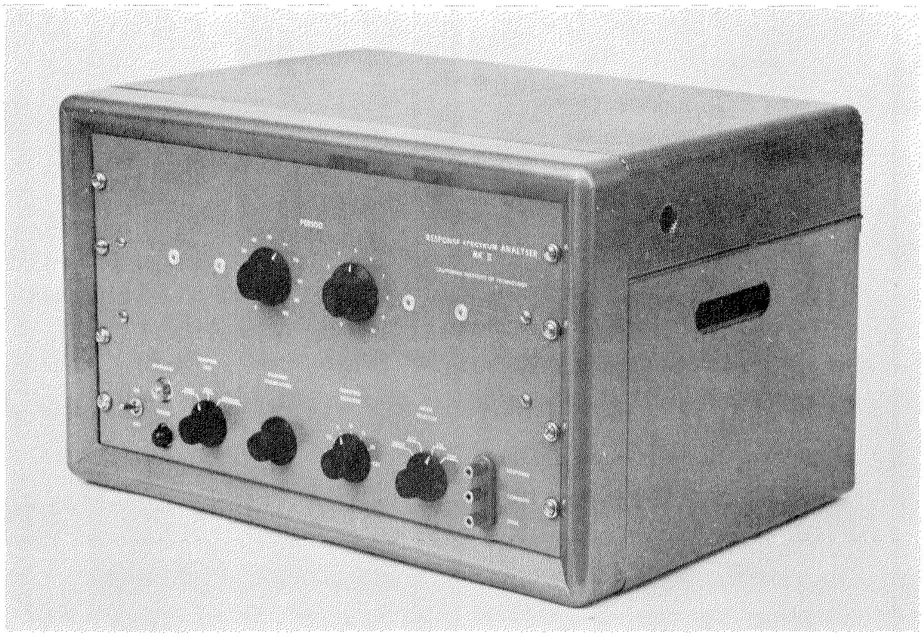


Fig. 5 Front Panel View of Spectrum Analyzer

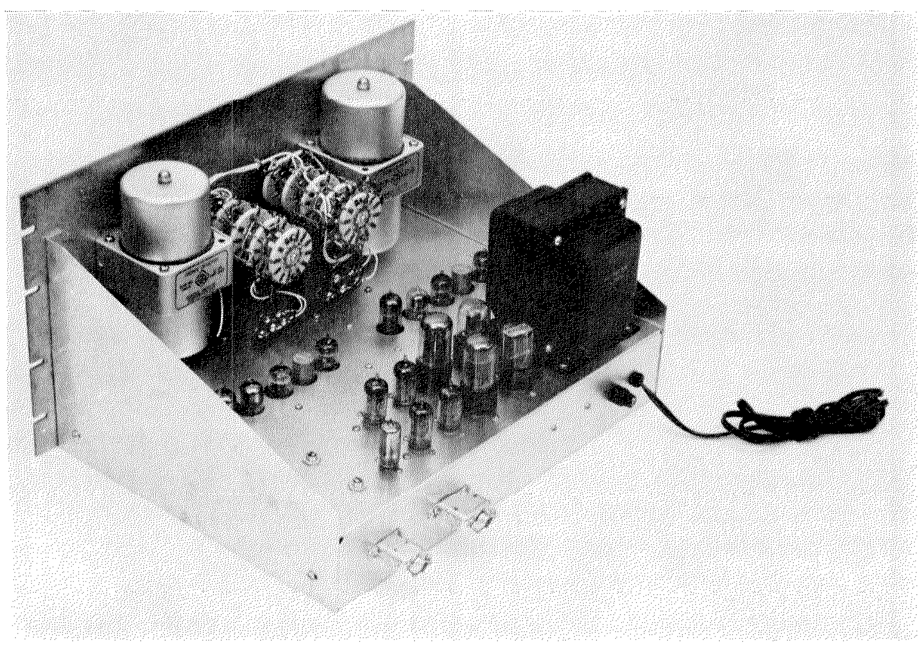


Fig. 6 Interior View of Spectrum Analyzer

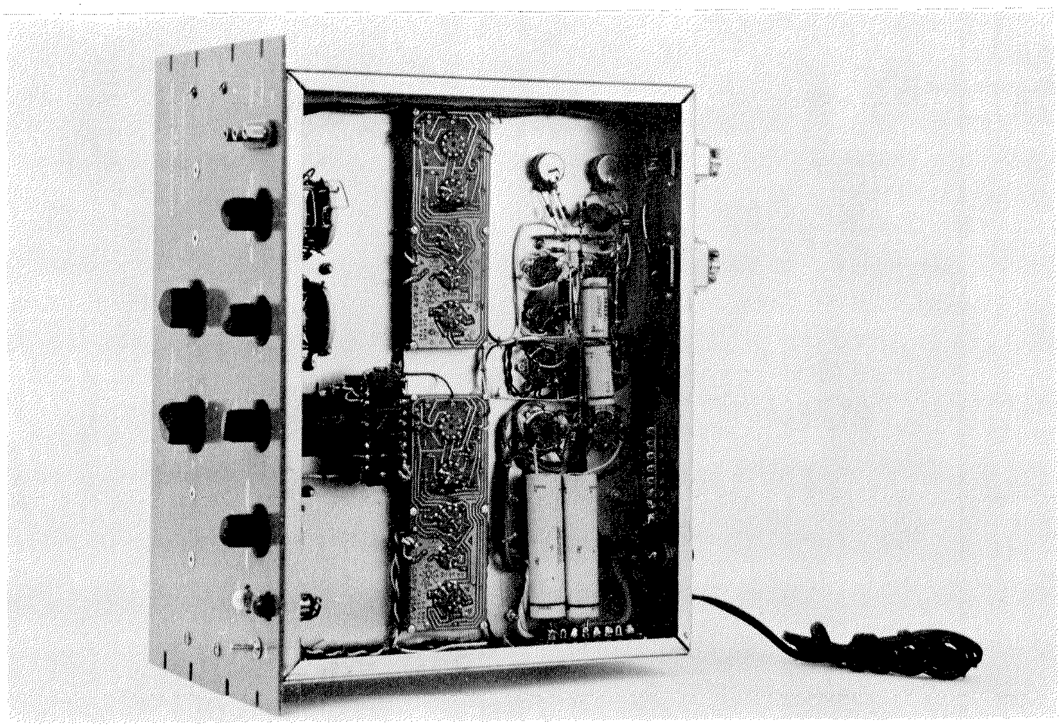


Fig. 7   Bottom View of Spectrum Analyzer

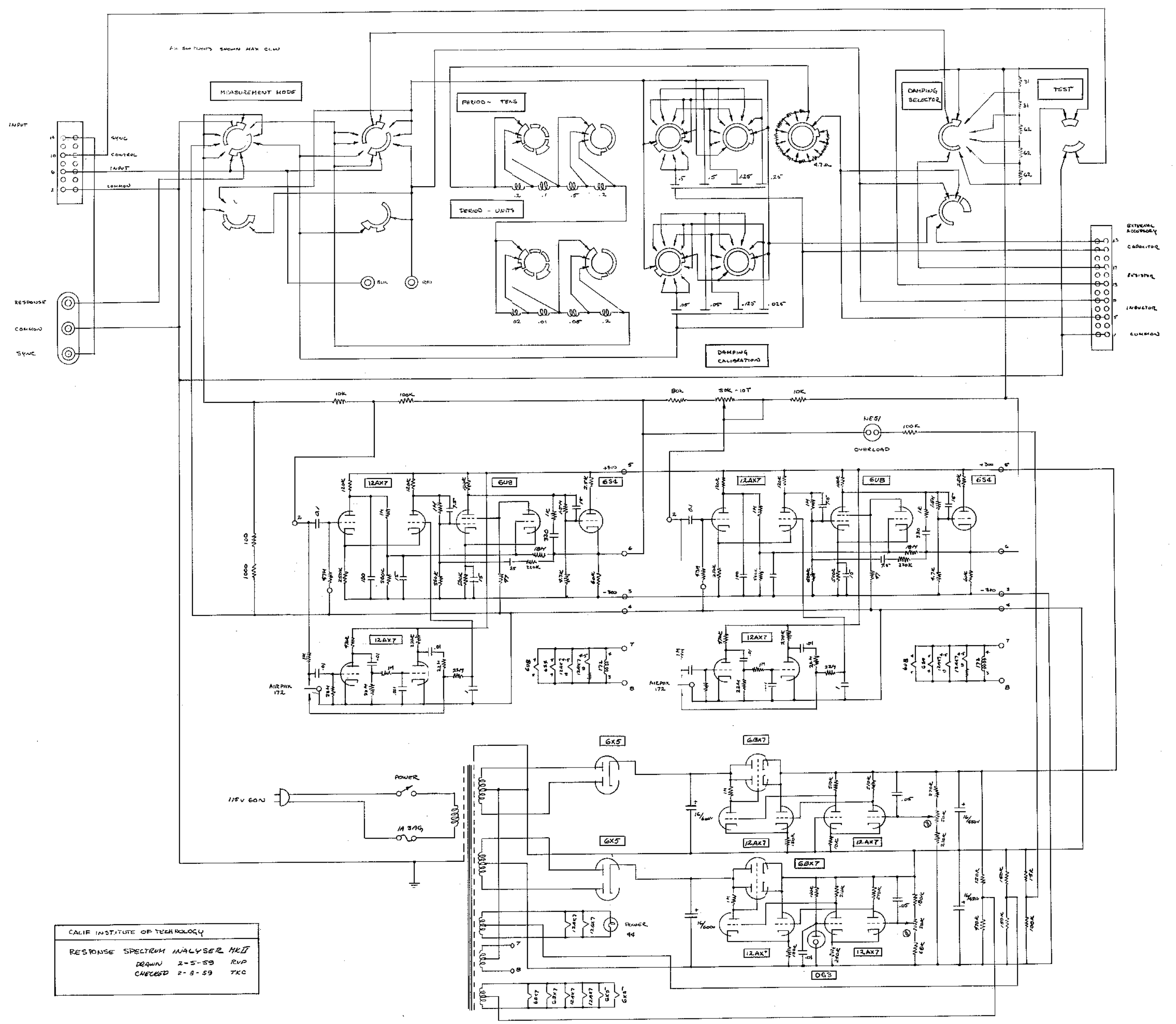


Fig. 8. Complete Electrical Circuit Diagram for the Spectrum Analyzer

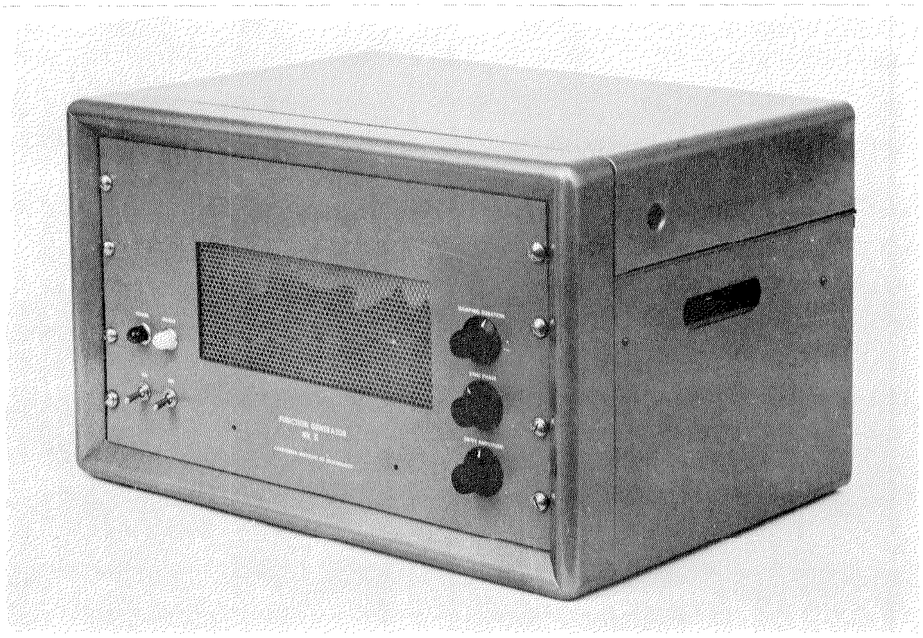


Fig. 9 Front Panel View of Function Generator

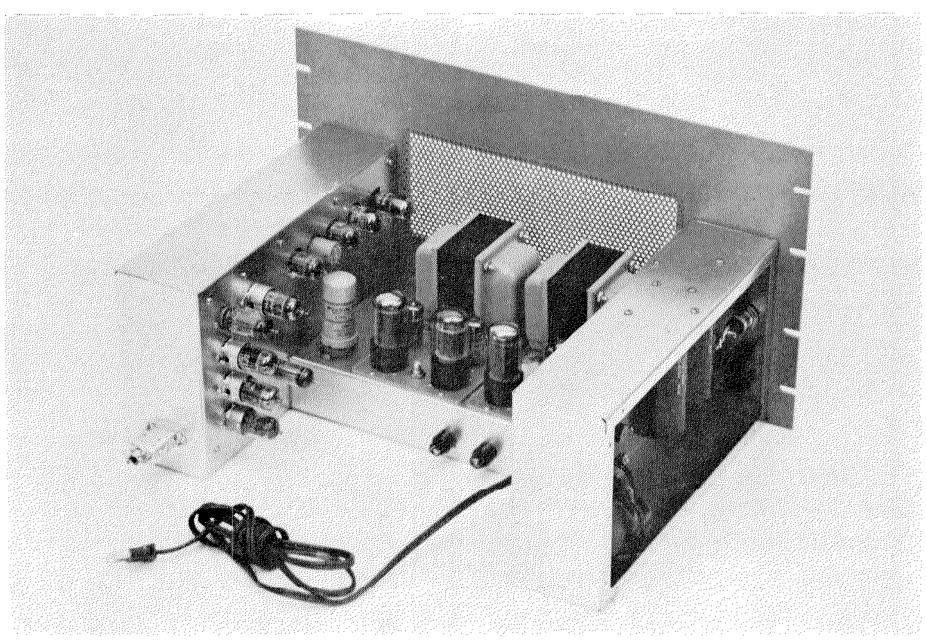


Fig. 10 Interior View of Function Generator

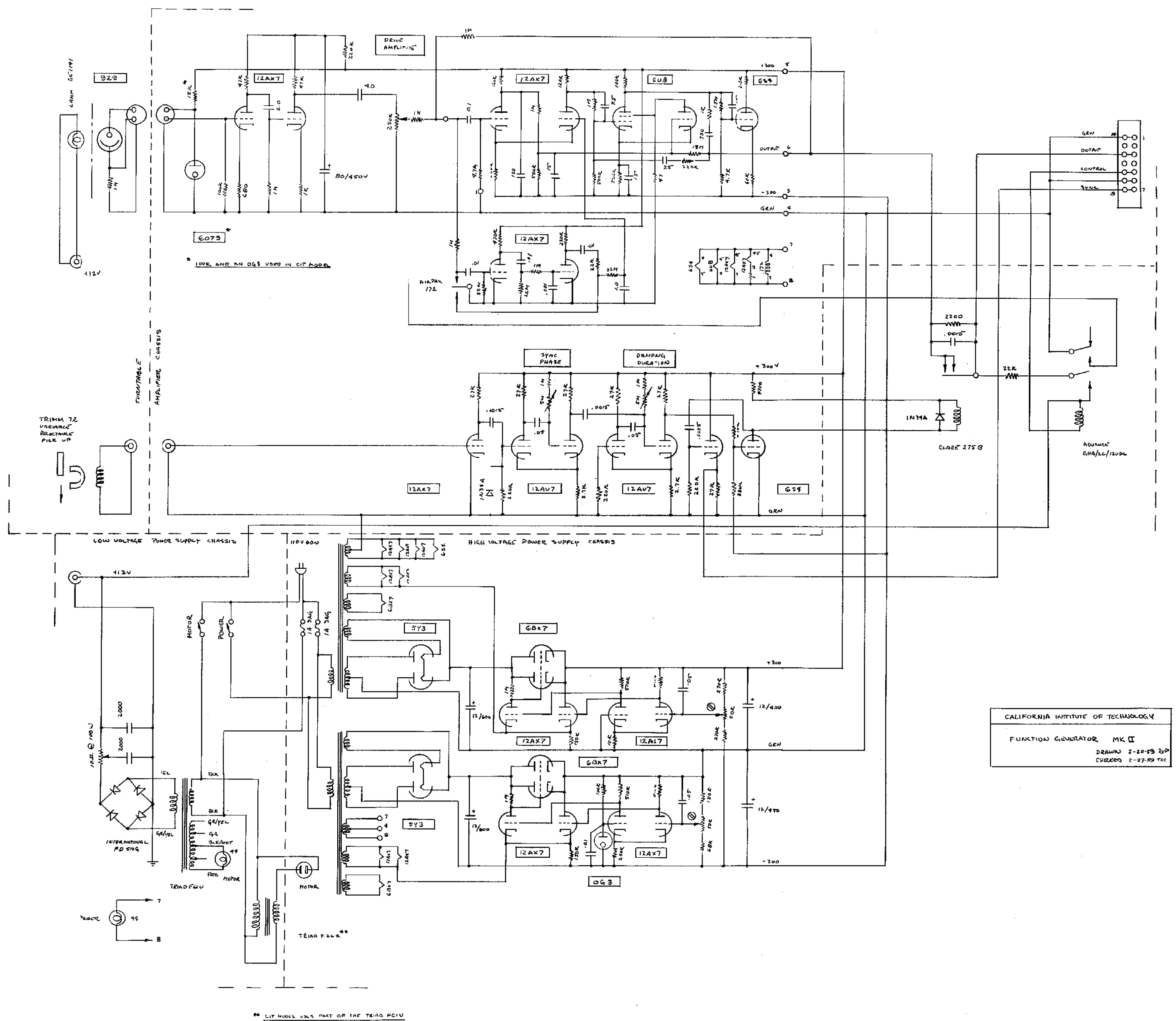


Fig. 11. Complete Circuit Diagram of Function Generator

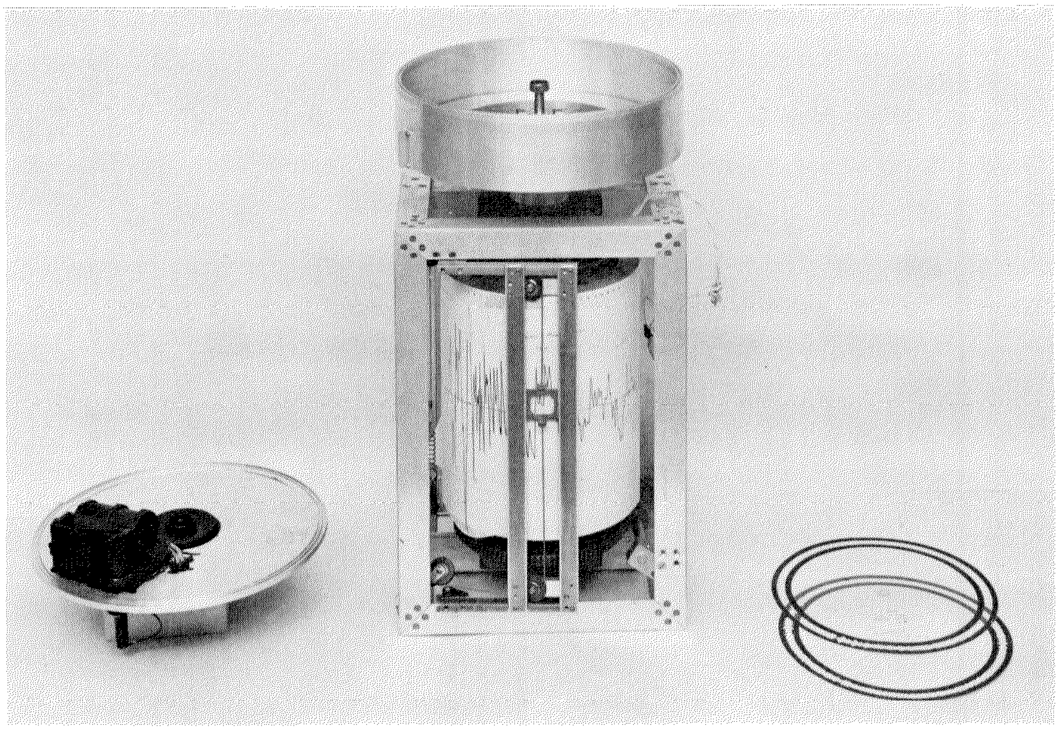


Fig. 12 Plotting Table for Function Generator Film Disk Record

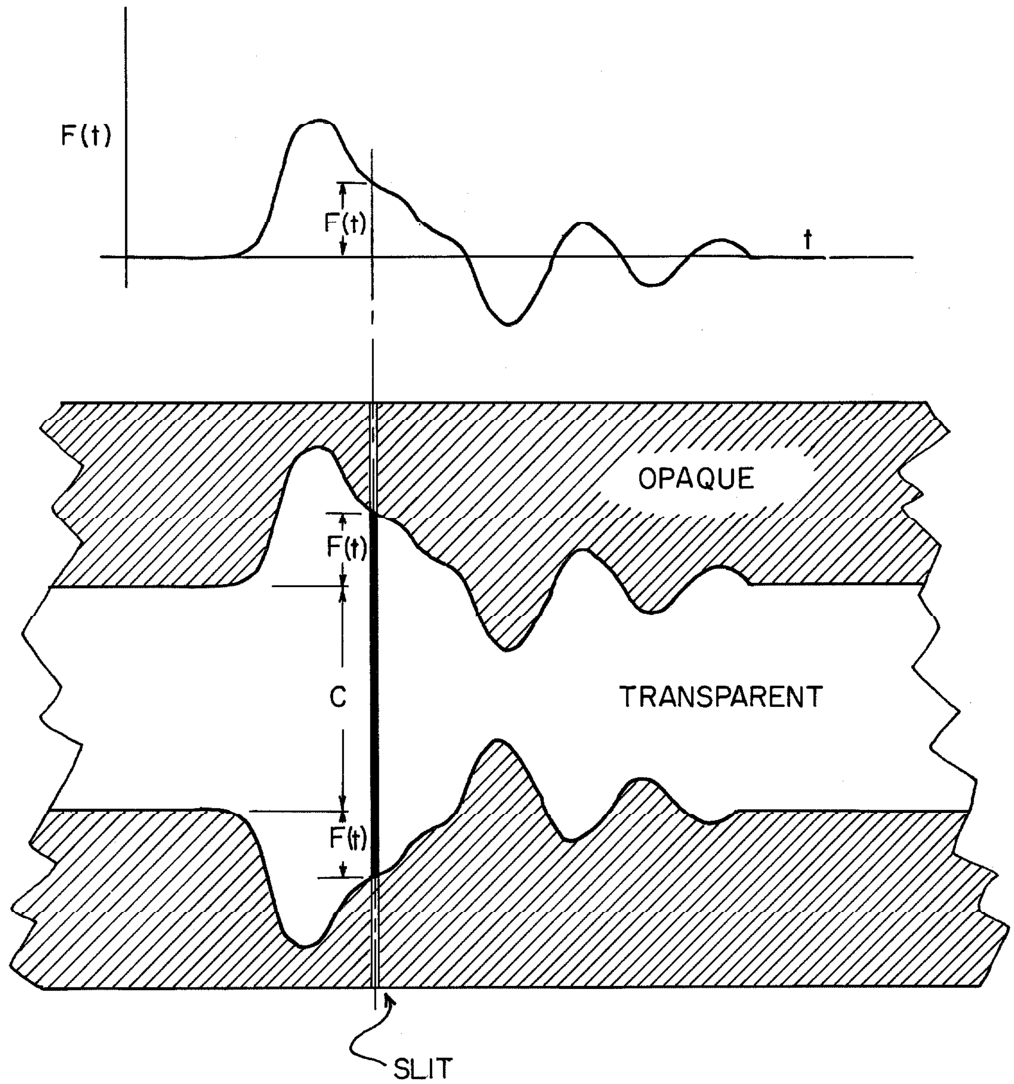


Fig. 13 Transformation of Input Function to Variable-Width Film Trace

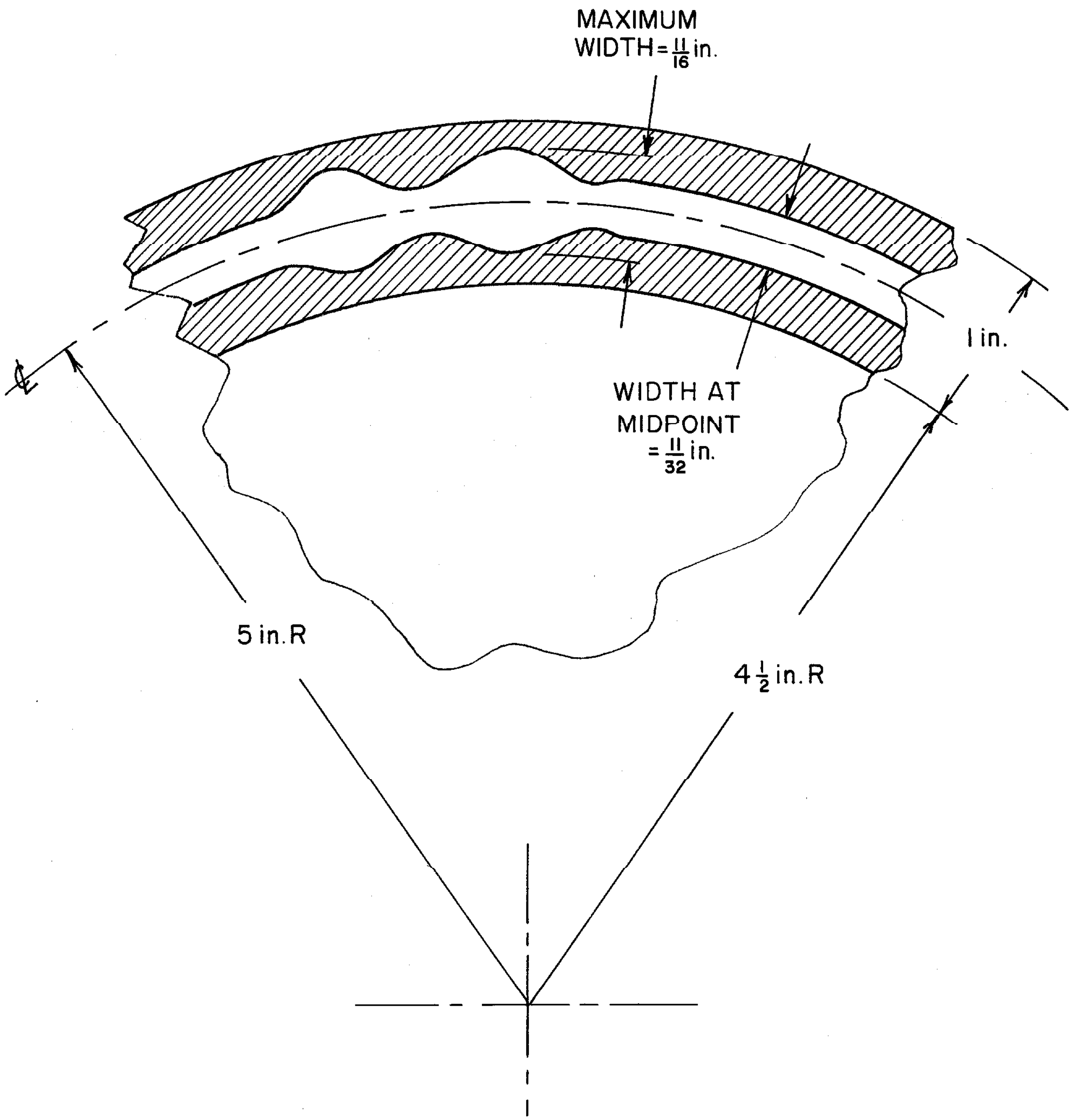


Fig. 14 Dimensions of the Standard Film Disk Record for the Function Generator

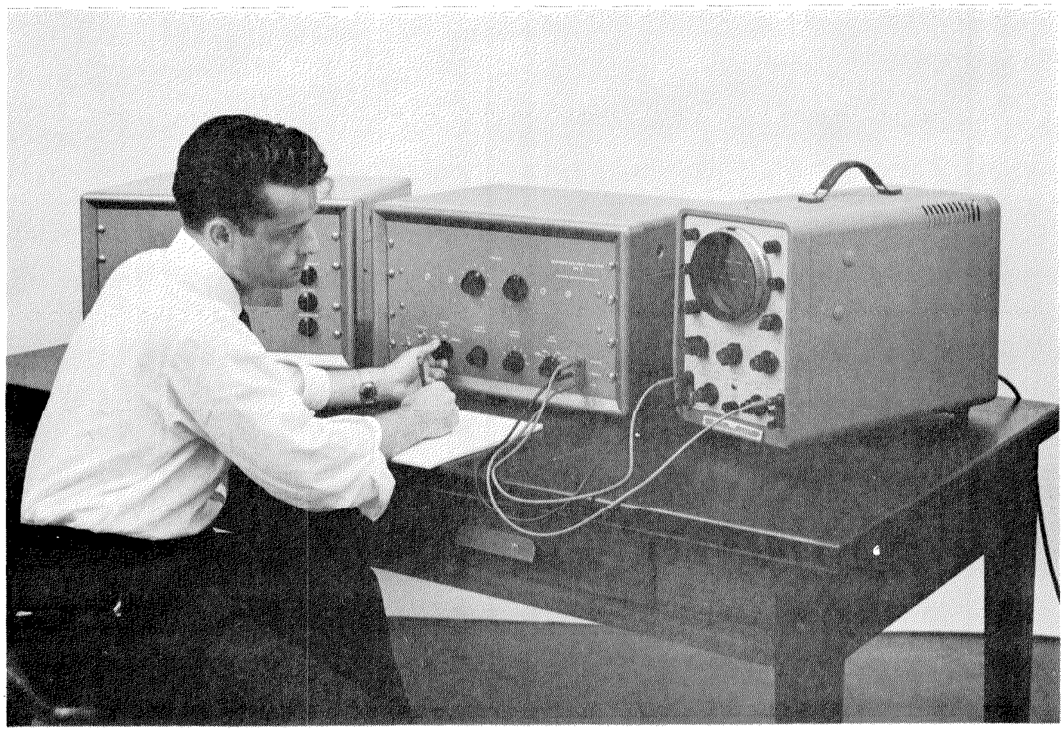
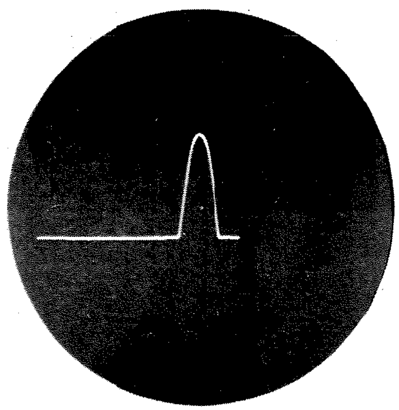
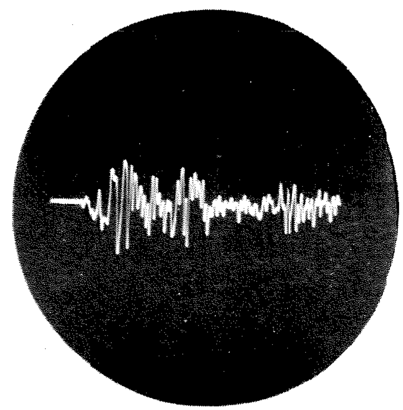


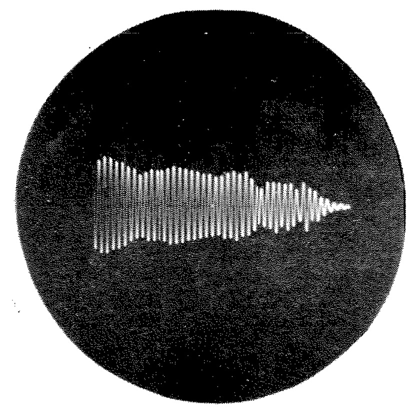
Fig. 15 Photograph of Complete Function Generator-Response Spectrum-Oscilloscope System



(a) Half-Sine Test Function



(b) Input Earthquake Acceleration First 10 Sec. of El Centro May 18, 1940, NS.



(c) Spectrum Analyzer Response to Earth-Quake of Fig. 16 (b)

Fig. 16 Typical Input and Response Records as Photographed from the Screen of the Cathode Ray Oscillograph

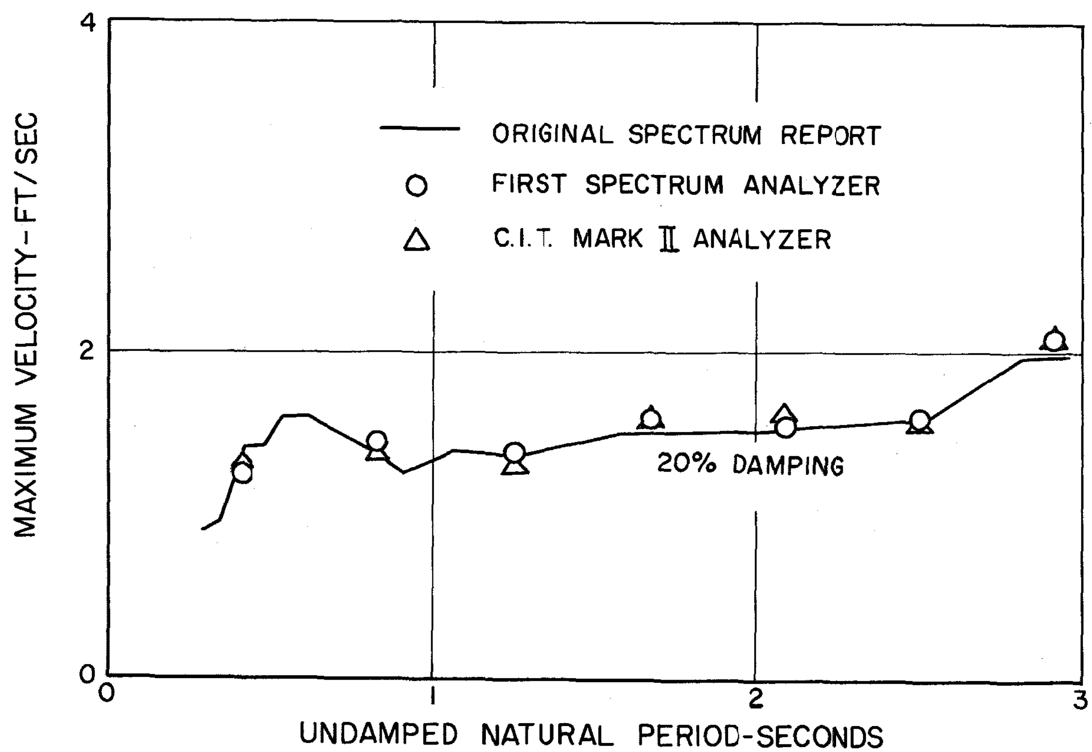


Fig. 17. Comparison of Response Spectrum Determinations for Earthquake Ground Motion

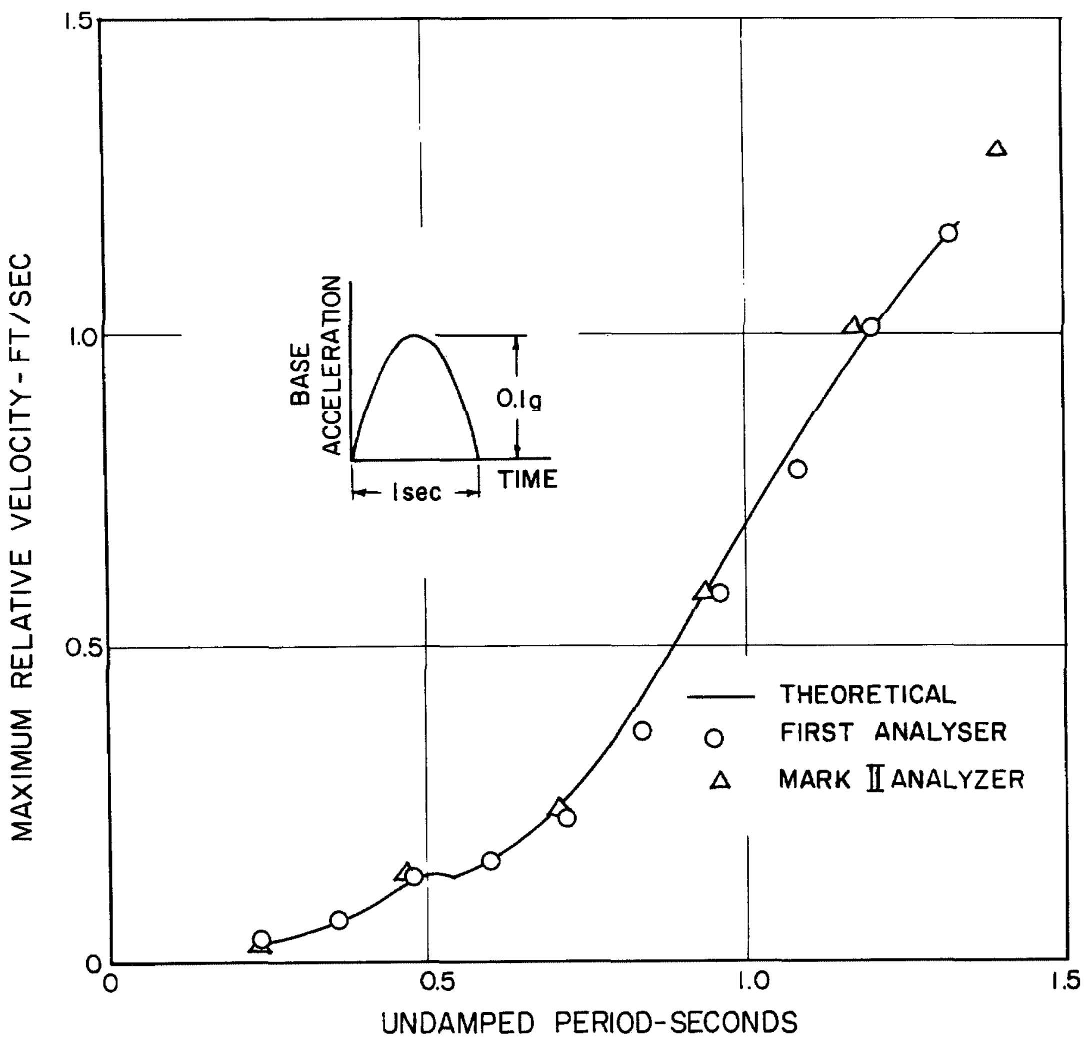


Fig. 18 Comparison of Response Spectrum Determinations for Half-Sine Pulse